APRIL/MAY 2024

23PMA12 — REAL ANALYSIS – I

Shanmuga Industries Arts & Science College T.V.Malai

Time : Three hours Maximum : 75 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL the questions.

1. Define Total variation.
2. Define Bounded variation.
3. Define step function.
4. Define Riemann's condition.
5. List the sufficient Candidates for the existence of the Riemann's integral.
6. Define measure zero.
7. Define Cauchy Product of Series.
8. Define Infinite Product.
9. Define Boundedly Convergent.
10. Define Uniform Convergence.

SECTION B — ($5 \times 5 = 25$ marks)

Answer ALL the questions.

11. (a) If f is monotonic on [a, b], then prove that the set of discontinuities of f is countable.

Or

(b) Let f be continuous on $[a,b]$, Then show that f is of bounded variation on $[a,b]$ iff f can be expressed as the difference of two increasing functions.

12. (a) Prove that If $f \in R(\alpha)$ and if $g \in R(\alpha)$ on $[a,b]$ then $c_1 f + c_2 f \in R(\alpha)$ on $[a,b]$ (for any two constants c_1 and c_2) and we have $\int_a^b (c_1 f + c_2 g)\, d\alpha = c_1 \int_a^b f d\alpha + c_1 \int_a^b g\, dx$.

Or

(b) Assume that α increasing on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then show that $f^2 \in (\alpha)$ on $[a,b]$.

13. (a) If f is continuous on $[a,b]$ and if α is of bounded variation on $[a,b]$, then show that $f \in R(\alpha)$ on $[a,b]$.

Or

20. Let α be of bounded variation on $[a, b]$ and assume that $\Sigma f_n(x) = f(x)$ (uniformly on [a, b]), where each function is real-valued function such that $f_n \in R(\alpha)\,[a,b]$. Then prove the following

(a) $f \in R(\alpha)$ on $[a, b]$.

(b) $\int_a^x \sum_{n=1}^{\infty} f_n(t)\, d\alpha(t) = \sum_{n=1}^{\infty} \int_a^x f_n(t)\, d\alpha(t)$ (Uniformly on $[a, b]$).

(b) State and prove that second mean value theorem.

14. (a) Let Σa_n and Σb_n be the absolutely convergent series with sums A and B, respectively. Let f be the double sequence defined by the equation. $f(m,n) = a_m b_n$, if $(m,n) \in z^+ \times z^+$. Then show that $\Sigma_{m,n} f(m,n)$ Convergence absolutely and has the sum AB.

Or

(b) Assume that each $a_n \geq 0$. Then prove that the product $\Pi(1 - a_n)$ converges if and only if the series Σa_n converges.

15. (a) Stet and prove weierstrass M-Test.

Or

(b) Assume that $\lim_{n\to\infty} f_n = f$ on $[a,b]$. If $g \in R$ on $[a,b]$ define $h(x) = \int_a^x f(+)\, g(+)\, dt, h_n(x) = \int_a^x f(+g(+)\, dt,$ if $x \in [a,b]$. Then prove that $h_n \to$ h uniformly on $[a,b]$.

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. Let f be of bounded variation on $[a,b]$ and assume that $c \in [a,b]$. Then prove that f is of bounded variation on $[a,b]$ and on $[c,b]$ and we have,

Prove that $V_f(a,b) = V_f(a,c) + V_f(c,b)$.

17. If f has continuous derivative f' on $[a,b]$, then we have $\sum_{a<n\le b} f(n) = \int_a^b f'(x) + \int_a^b f'(x)((x))dx$ $+ f(a)((a)) - f(b)((b))$, where $((x)) = x - [x]$, when a and b are integers, this becomes $\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\left(x - [x] - \frac{1}{2}\right)dx +$ $\frac{f(a)+f(b)}{2}$ (Euler's Summatioin formula.

18. Assume that α is of bounded variation on $[a,b]$. Let $v(x)$ denote the total variation of α on $[a,x]$ if $a < x \le b$, and let $V(a) = 0$. Let f be defined and bounded on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$, then prove that $f \in R(v)$ on $[a,b]$.

19. Let $\sum f(m,n)$ be a given double series and let g be an arrangement of the double sequence of into a sequence G, Then prove the following

(a) $\sum G(n)$ converges absolutely if and only if $\sum f(m,n)$ converges absoultely. Assuming that $\sum f(m,n)$ does converge absolutely with sum S, we have furtier :

(b) $\sum_{n=1}^{\infty} G(n) = s$.

(c) $\sum_{n=1}^{\infty} f(m,n)$ and $\sum_{n=1}^{\infty} f(m,n)$ both converge absolutely.

(d) If $A_m = \sum_{n=1}^{\infty} f(m,n)$ and $B_n = \sum_{m=1}^{\infty} f(m,n)$, both series ΣA_m and ΣB_n converge absolutely and both have sum S. That is , $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = S$.